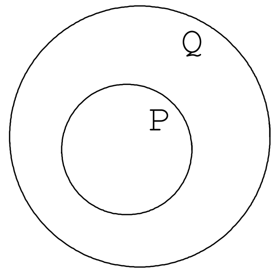
[10-09-08-T12] REV

Sets and implication

Consider the set Q of things to which the predicate "is mortal" is correctly ascribed. And consider the set P of things to which the predicate "is human" correctly applies. Finally, let x be a thing. The proposition q, "x is mortal", is true just in case $x \in Q$. And, the proposition p, "x is human", is true for exactly those things that are in P. Certainly, $P \subseteq Q$. We represent this with a Venn Diagram,



We say "p is a sufficient condition for q" because every x in P must be in Q. To know that x is in Q, it suffices to know that x is in P. Once I know a thing is human, I know it is mortal. Notice that q is not a sufficient condition for p, because membership in Q does not guarantee membership in P.

We say "q is a necessary condition for p" because every x in P is necessarily in Q. Knowledge that x is not in Q is knowledge that x is not in P, because membership in Q is necessary for membership in P.

We can summarize the necessary and sufficient conditions with regard to p and q by writing "p implies q", symbolically "$p \Longrightarrow q$". This is exactly what one expects, because we all know it is true that "if a thing is human, then the thing is mortal".

■ Truth table for $p \Longrightarrow q$.

$p \Longrightarrow q$ is a proposition, albeit a compound proposition. If a proposition is not false, then it must be true, because there are only two possibilities: true, false.

We wish to complete the following table:

p	$\Longrightarrow$	q
T	□	T
T	□	F
F	□	T
F	□	F

Line 1 is not controversial, certainly $\begin{array}{ccc} \boldsymbol{p} & \Longrightarrow & \boldsymbol{q} \\ \boldsymbol{T} & \boldsymbol{T} & \boldsymbol{T} \end{array}$.

Line 2 is also as we would expect, because if there is an object in P but not in Q, then it is certainly false that $P \subseteq Q$ whence $p \Longrightarrow q$ is false. So $\begin{array}{ccc} \boldsymbol{p} & \Longrightarrow & \boldsymbol{q} \\ \boldsymbol{T} & \boldsymbol{F} & \boldsymbol{F} \end{array}$.

As for line 3, my dog Miley is mortal and not human, so Miley is a member of Q but not of P. The existence of an element in Q and not in P does not make $P \subseteq Q$ false so is does not make $p \Longrightarrow q$ false.

Therefore $p \Longrightarrow q$ must be true (because it is not false). That is, $\begin{matrix} \boldsymbol{p} & \boldsymbol{\Longrightarrow} & \boldsymbol{q} \\ \boldsymbol{F} & \boldsymbol{T} & \boldsymbol{T} \end{matrix}$.

To understand line 4, think of a rock. The rock is not human and the rock is not mortal. The rock is an element that is neither in P nor in Q. The existence of such an element does not make $P \subseteq Q$ false, thus it does not make false the proposition $p \Longrightarrow q$, so the proposition must be true (because it is not false). That is, $\begin{matrix} \boldsymbol{p} & \boldsymbol{\Longrightarrow} & \boldsymbol{q} \\ \boldsymbol{F} & \boldsymbol{T} & \boldsymbol{F} \end{matrix}$.

We now have the complete truth table for $p \Longrightarrow q$. It is

p	$\boldsymbol{\Longrightarrow}$	***q***
T	***T***	***T***
T	***F***	***F***
F	***T***	***T***
F	***T***	***F***

■ Another way to understand the truth table for $p \Longrightarrow q$

Ask yourself how you would argue a statement of the form $p \Longrightarrow q$ is false. Constructing a counterexample would do the job. Now, only an example in which p is true and q is false will be a counterexample. That is line 2 of the truth table. You know better than to argue against a statement of the form $p \Longrightarrow q$ by making up examples that match lines 1, 3, or 4 of the truth table. So, you already knew the truth table for $p \Longrightarrow q$ even before we started this discussion!